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Separation of variables in the Hamilton–Jacobi equation for non-conservative systems

F Cantrijn†
Instituut voor Theoretische Mechanica, Rijksuniversiteit Gent, B-9000 Gent, Belgium

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Abstract. Extending the method of Havas for conservative systems, the separability of the Hamilton–Jacobi equation is investigated for mechanical systems described by a time-dependent Hamiltonian, including systems possessing a velocity-dependent potential energy. It is shown that for n degrees of freedom there exist $n+1$ different types of separable systems, of which the corresponding Hamiltonians are derived after constructing the separated differential equations. Herewith a more profound and systematic approach is given to the results of Iarov-Iarovoi, which have been obtained on a more intuitive basis.

1. Introduction

In 1963, Iarov-Iarovoi solved the problem of determining all Hamiltonians of the form

$$H \equiv \tfrac{1}{2} \sum_{i,j=1}^{n} g^{ij}(q_1, \ldots, q_n, t) p_i p_j + \sum_{i=1}^{n} g^{i}(q_1, \ldots, q_n, t) p_i + \tfrac{1}{2} g^{0}(q_1, \ldots, q_n, t) + V(q_1, \ldots, q_n, t), \qquad (1)$$

where $g^{ij} \equiv g^{ji}$, for which the corresponding Hamilton–Jacobi equation‡

$$\tfrac{1}{2} \sum_{i,j=1}^{n} g^{ij}(q,t) \frac{\partial W}{\partial q_i} \frac{\partial W}{\partial q_j} + \sum_{i=1}^{n} g^{i}(q,t) \frac{\partial W}{\partial q_i} + \tfrac{1}{2} g^{0}(q,t) + V(q,t) + \frac{\partial W}{\partial t} = 0 \qquad (2)$$

is separable. In spite of the briefness of the method developed by Iarov-Iarovoi (1963), there is, however, a certain lack of clarity in some of his arguments. Moreover a solution is postulated which, afterwards, turns out to be the most general one. Finally some characteristics of the functions occurring in the Hamiltonian under consideration, merit a closer inspection. The main purpose of the present article is to give a more systematic approach to the problem.

In an article on the separation of variables in the Hamilton–Jacobi, Schrödinger and related equations, Havas (1975) derived all types of time-independent Hamiltonians, without linear terms in the momenta, for which the Hamilton–Jacobi equation is separable. His work was essentially based on the results obtained by Levi-Civita (1904), who proved the existence of $n+1$ types of separable systems in n dimensions, and by Dall'Acqua (1912) and Burgatti (1911), who gave the general form of the

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† Aspirant bij het Nationaal Fonds voor Wetenschappelijk Onderzoek, Belgium.

‡ Here and in the following we shall write q for the collection of the q_i and p for the collection of the p_i ($i = 1, \ldots, n$).

separated differential equations. (For another approach to the time-independent problem we also refer to some articles of Agostinelli (1936), (1937).)

The generalization to non-conservative systems first appears in a paper of Forbat (1944). He deduced the conditions satisfied by a Hamiltonian of the form (1) for which separation of the variables in (2) is possible:

$$\frac{\partial H}{\partial p_i}\left(\frac{\partial H}{\partial p_j}\frac{\partial^2 H}{\partial q_i\,\partial q_j}-\frac{\partial H}{\partial q_j}\frac{\partial^2 H}{\partial q_i\,\partial p_j}\right)$$

$$\equiv\frac{\partial H}{\partial q_i}\left(\frac{\partial H}{\partial p_j}\frac{\partial^2 H}{\partial p_i\,\partial q_j}-\frac{\partial H}{\partial q_j}\frac{\partial^2 H}{\partial p_i\,\partial p_j}\right),\qquad i,j=1,2,\dots,n,\ (i\neq j)\qquad (\mathrm{C}_1)$$

$$\frac{\partial H}{\partial p_i}\frac{\partial^2 H}{\partial q_i\,\partial t}\equiv\frac{\partial H}{\partial q_i}\frac{\partial^2 H}{\partial p_i\,\partial t},\qquad i=1,2,\dots,n.\qquad (\mathrm{C}_2)$$

These conditions are a generalization of those deduced by Levi-Civita (1904) for time-independent Hamiltonians. They are to be satisfied identically.

Forbat further treated the special case in which H contains no linear terms in the momenta ($g^i\equiv 0$ for all i) and $\frac{1}{2}g^0+V$ is supposed to depend on all the generalized coordinates. Starting from the conditions (C_1) and (C_2) our treatment will be an extension of the one followed by Dall'Acqua (1912) and Havas (1975).

In § 2 we shall derive some further properties of a Hamiltonian satisfying (C_1) and (C_2). In § 3 the results so obtained will be used for constructing the general form of the separated differential equations. A straightforward calculation will then finally lead to a necessary and sufficient condition for separability, giving the general form of the functions g^{ij}, g^i and $\frac{1}{2}g^0+V$, which also occur in the article of Iarov-Iarovoi (1963). Along with some general remarks, in the last section we shall also make a comparison with the special case treated by Forbat (1944). The Hamiltonian (1) describes general non-conservative systems, including those having a velocity-dependent potential energy. In the latter case it is known that the potential energy may only depend linearly on the generalized velocities (Gantmacher 1970). The velocity-independent part of the potential energy is denoted by V.

Restricting ourselves to mechanical systems, all functions appearing in (1) are supposed to be continuous and sufficiently differentiable in an appropriate domain.

2. Preliminary calculations

Consider a mechanical system, possessing a Hamiltonian of the form (1). We shall sometimes use the abbreviation:

$$H\equiv H_2+H_1+H_0,$$

where

$$H_2\equiv\frac{1}{2}\sum_{i,j=1}^{n}g^{ij}p_ip_j;\qquad H_1\equiv\sum_{i=1}^{n}g^ip_i;\qquad H_0\equiv\frac{1}{2}g^0+V.$$

We assume that $\partial H/\partial p_i\not\equiv 0$ for all $i=1,2,\dots,n$, for otherwise the number of degrees of freedom could immediately be diminished. Suppose the Hamilton–Jacobi equation (2) is separable. The conditions (C_1) and (C_2) are thus satisfied and there must exist a complete integral of the form

$$W(q, c, t) \equiv W_0(c, t) + \sum_{i=1}^{n} W_i(q_i, c), \tag{3}$$

where c is a set of n real independent arbitrary constants $c_1, c_2, \ldots, c_n$.

Our aim is to construct such a complete integral, following an analogous method such as the one used by Dall'Acqua (1912). In both identities (C_1) and (C_1), the left-hand side is divisible by $\partial H/\partial p_i$† and, consequently, so must be the right-hand side. We now divide the coordinates into two disjunct sets, called coordinates of the first and of the second kind, respectively,

$$I_1 = \{q_i \colon \partial H/\partial q_i \text{ is divisible by } \partial H/\partial p_i \text{ or } \partial H/\partial q_i \equiv 0\},$$

$$I_2 = \{q_r \colon \partial H/\partial q_r \not\equiv 0 \text{ and } \partial H/\partial q_r \text{ is not divisible by } \partial H/\partial p_r\}.$$

For each $q_i \in I_1$ there exists a function $N_i(q, p, t)$, polynomial in the momenta, such that

$$\frac{\partial H}{\partial q_i} \equiv \frac{\partial H}{\partial p_i} N_i(q, p, t). \tag{4}$$

It follows from (4) that N_i can be written as

$$N_i \equiv N_i^{(1)} + N_i^{(0)},$$

with $N_i^{(1)}(q, p, t)$ a homogeneous linear function in the momenta and $N_i^{(0)}(q, t)$ independent of the momenta. Splitting up (4) we obtain:

$$\frac{\partial H_2}{\partial q_i} \equiv \frac{\partial H_2}{\partial p_i} N_i^{(1)}, \tag{4a}$$

$$\frac{\partial H_1}{\partial q_i} \equiv \frac{\partial H_2}{\partial p_i} N_i^{(0)} + g^i N_i^{(1)}, \qquad q_i \in I_1, \tag{4b}$$

$$\frac{\partial H_0}{\partial q_i} \equiv g^i N_i^{(0)}. \tag{4c}$$

For each $q_r \in I_2$, it follows from (C_1) and (C_2) that the factors on the right-hand sides, different from $\partial H/\partial q_r$, must be divisible by $\partial H/\partial p_r$. Consequently, there must exist a function $K_r(q, p, t)$ and, for each $s (s \neq r)$, a function $M_{sr}(q, p, t)$, which are polynomial in the momenta and satisfy the following identities:

$$\frac{\partial H}{\partial p_s} \frac{\partial^2 H}{\partial q_s\, \partial p_r} - \frac{\partial H}{\partial q_s} \frac{\partial^2 H}{\partial p_s\, \partial p_r} \equiv \frac{\partial H}{\partial p_r} M_{sr}, \qquad s \neq r,\ q_r \in I_2, \tag{5}$$

$$\frac{\partial^2 H}{\partial p_r\, \partial t} \equiv \frac{\partial H}{\partial p_r} K_r. \tag{6}$$

One can easily check that K_r has to be independent of the momenta, i.e. $K_r \equiv K_r^{(0)}(q, t)$, and that M_{sr} can be written as

$$M_{sr} \equiv M_{sr}^{(1)} + M_{sr}^{(0)},$$

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† In this paper, a function $f(q, p, t)$ is said to be divisible by $\partial H/\partial p_i$ if and only if there exists a function $g(q, p, t)$ being polynomial in the momenta, such that $f(q, p, t) \equiv (\partial H/\partial p_i) g(q, p, t)$.

with $M_{sr}^{(1)}(q,p,t)$ homogeneous linear in the momenta and $M_{sr}^{(0)}(q,t)$ independent of the momenta. Both identities (5) and (6) can now be split up into respectively

$$\frac{\partial H_2}{\partial p_s}\frac{\partial^2 H_2}{\partial q_s\,\partial p_r}-\frac{\partial H_2}{\partial q_s}g^{rs}\equiv\frac{\partial H_2}{\partial p_r}M_{sr}^{(1)}, \tag{5a}$$

$$\frac{\partial H_2}{\partial p_s}\frac{\partial g^r}{\partial q_s}+g^s\frac{\partial^2 H_2}{\partial q_s\,\partial p_r}-\frac{\partial H_1}{\partial q_s}g^{rs}\equiv\frac{\partial H_2}{\partial p_r}M_{sr}^{(0)}+g^rM_{sr}^{(1)}, \tag{5b}$$

$$g^s\frac{\partial g^r}{\partial q_s}-\frac{\partial H_0}{\partial q_s}g^{rs}\equiv g^rM_{sr}^{(0)}, \tag{5c}$$

and

$$\frac{\partial^2 H_2}{\partial p_r\,\partial t}\equiv\frac{\partial H_2}{\partial p_r}K_r^{(0)}, \tag{6a}$$

$$\frac{\partial g^r}{\partial t}\equiv g^rK_r^{(0)}. \tag{6b}$$

We shall first derive an explicit expression for the functions M_{sr}. Differentiation of (5) with respect to p_r yields

$$\frac{\partial H}{\partial p_s}\frac{\partial g^{rr}}{\partial q_s}\equiv g^{rr}M_{sr}+\frac{\partial H}{\partial p_r}\frac{\partial M_{sr}}{\partial p_r}, \tag{7}$$

and, after a second differentiation with respect to p_r:

$$g^{rs}\frac{\partial g^{rr}}{\partial q_s}\equiv 2g^{rr}\frac{\partial M_{sr}}{\partial p_r},$$

or

$$\frac{\partial M_{sr}}{\partial p_r}\equiv\frac{g^{rs}}{2g^{rr}}\frac{\partial g^{rr}}{\partial q_s}. \tag{8}$$

It may be noticed here, for justifying this last step, that the functions g^{jj} ($j=1,2,\ldots,n$) vanish nowhere in the relevant domain. This arises from some considerations about the term H_2 in H. In fact, for a mechanical system, this term ought to be positive definite, for it represents the quadratic part of the kinetic energy (expressed in the momenta). It then follows from Sylvester's inequalities that none of the diagonal elements g^{jj} of the symmetric square matrix (g^{ij}) may vanish (Gantmacher 1970). (They even have to be strictly positive.)

Substitution of (8) into (7) now gives rise to the following result:

$$M_{sr}\equiv\frac{1}{2(g^{rr})^2}\frac{\partial g^{rr}}{\partial q_s}\left(2g^{rr}\frac{\partial H}{\partial p_s}-g^{rs}\frac{\partial H}{\partial p_r}\right),\qquad q_r\in I_2,\ s\neq r, \tag{9}$$

and so:

$$M_{sr}^{(1)}\equiv\frac{1}{2(g^{rr})^2}\frac{\partial g^{rr}}{\partial q_s}\left(2g^{rr}\frac{\partial H_2}{\partial p_s}-g^{rs}\frac{\partial H_2}{\partial p_r}\right), \tag{9a}$$

$$M_{sr}^{(0)}\equiv\frac{1}{2(g^{rr})^2}\frac{\partial g^{rr}}{\partial q_s}(2g^{rr}g^s-g^{rs}g^r). \tag{9b}$$

Taking into account (5) and (6), the identities (C_1) and (C_2) can be written, for each $q_r \in I_2$, as

$$\frac{\partial H}{\partial p_s}\frac{\partial^2 H}{\partial q_r\,\partial q_s}-\frac{\partial H}{\partial q_s}\frac{\partial^2 H}{\partial q_r\,\partial p_s}\equiv\frac{\partial H}{\partial q_r}M_{sr}, \qquad (s\neq r) \tag{10a}$$

$$\frac{\partial^2 H}{\partial q_r\,\partial t}\equiv\frac{\partial H}{\partial q_r}K_r^{(0)}. \tag{10b}$$

Differentiation of (5) with respect to q_r and of (10a) with respect to p_r gives, after subtraction of both results,

$$\frac{\partial H}{\partial p_r}\frac{\partial M_{sr}}{\partial q_r}\equiv\frac{\partial H}{\partial q_r}\frac{\partial M_{sr}}{\partial p_r}+2\frac{\partial^2 H}{\partial q_r\,\partial p_s}\frac{\partial^2 H}{\partial q_s\,\partial p_r}-2g^{rs}\frac{\partial^2 H}{\partial q_r\,\partial q_s}. \tag{11}$$

Calculating $\partial^2 H/\partial q_r\,\partial q_s$ and $\partial^2 H/\partial q_s\,\partial p_r$ respectively from (10a) and (5) and substituting the results into (11), we obtain after a straightforward calculation (using (8) and (9))

$$\begin{aligned}\frac{\partial M_{sr}}{\partial q_r}\frac{\partial H}{\partial p_s}\frac{\partial H}{\partial p_r}\equiv&-\frac{3}{2}\frac{g^{rs}}{g^{rr}}\frac{\partial g^{rr}}{\partial q_s}\frac{\partial H}{\partial q_r}\frac{\partial H}{\partial p_s}+\left(\frac{g^{rs}}{g^{rr}}\right)^2\frac{\partial g^{rr}}{\partial q_s}\frac{\partial H}{\partial q_r}\frac{\partial H}{\partial p_r}\\&+\frac{2}{g^{rr}}\frac{\partial g^{rr}}{\partial q_s}\frac{\partial^2 H}{\partial q_r\,\partial p_s}\frac{\partial H}{\partial p_r}\frac{\partial H}{\partial p_s}-\frac{g^{rs}}{(g^{rr})^2}\frac{\partial g^{rr}}{\partial q_s}\frac{\partial^2 H}{\partial q_r\,\partial p_s}\left(\frac{\partial H}{\partial p_r}\right)^2.\end{aligned} \tag{12}$$

This must hold for all $s\,(s\neq r)$, whenever q_r is a variable of the second kind.

If $\partial H/\partial p_s$ is not divisible by $\partial H/\partial p_r$, it follows immediately from (12) that the first term on the right-hand side of this identity must vanish, since then it is the only one which is not divisible by $\partial H/\partial p_r$. This clearly means that

$$g^{rs}\frac{\partial g^{rr}}{\partial q_s}\equiv 0, \qquad q_r\in I_2,\ s\neq r, \tag{13}$$

taking into account that neither $\partial H/\partial q_r$ nor $\partial H/\partial p_s$ (by assumption) are identically zero.

It can be proved quite easily that (13) still holds when $\partial H/\partial p_s$ is divisible by $\partial H/\partial p_r$ for some particular $s\,(s\neq r)$. In that case, the left-hand side as well as the last two terms on the right-hand side of (12) are divisible by $(\partial H/\partial p_r)^2$. The sum of the first two terms on the right-hand side must vanish, yielding either (13) or

$$\frac{3}{2}\frac{\partial H}{\partial p_s}\equiv\frac{g^{rs}}{g^{rr}}\frac{\partial H}{\partial p_r}. \tag{14}$$

This, however, is consistent with (13), since differentiation of (14) with respect to p_r shows that in the latter case $g^{rs}\equiv 0$. With (13) M_{sr} becomes

$$M_{sr}\equiv\frac{1}{g^{rr}}\frac{\partial g^{rr}}{\partial q_s}\frac{\partial H}{\partial p_s},$$

and so, after re-arranging the terms in (5), we obtain

$$\frac{\partial H}{\partial p_s}\left(\frac{\partial^2 H}{\partial p_r\,\partial q_s}-\frac{1}{g^{rr}}\frac{\partial g^{rr}}{\partial q_s}\frac{\partial H}{\partial p_r}\right)\equiv\frac{\partial H}{\partial q_s}g^{rs}. \tag{15}$$

If $q_s\in I_2$, the right-hand side of (15) must vanish, for no factor is divisible by $\partial H/\partial p_s$. Since by definition of I_2, $\partial H/\partial q_s\not\equiv 0$, we must have

$$g^{rs}\equiv 0, \qquad q_r\in I_2,\ q_s\in I_2,\ r\neq s. \tag{16}$$

From (15) we also get in this case:

$$\frac{\partial^2 H}{\partial p_r\, \partial q_s} - \frac{1}{g^{rr}} \frac{\partial g^{rr}}{\partial q_s} \frac{\partial H}{\partial p_r} \equiv 0,$$

or, after division by g^{rr} (which is allowed according to the remark following (8))

$$\frac{1}{(g^{rr})^2} \left[g^{rr} \frac{\partial}{\partial q_s} \left(\frac{\partial H}{\partial p_r} \right) - \frac{\partial H}{\partial p_r} \frac{\partial g^{rr}}{\partial q_s} \right] \equiv 0,$$

and so

$$\frac{\partial}{\partial q_s} \left(\frac{1}{g^{rr}} \frac{\partial H}{\partial p_r} \right) \equiv 0, \qquad q_r \in I_2,\ q_s \in I_2,\ r \neq s. \tag{17}$$

Let us now consider the case that $q_r \in I_2$ and $q_s \in I_1$. If $g^{rs} \not\equiv 0$ it follows from (13) that

$$\frac{\partial g^{rr}}{\partial q_s} \equiv 0, \qquad q_r \in I_s,\ q_s \in I_1. \tag{18}$$

By (4) we also have:

$$\frac{\partial H}{\partial q_s} \equiv \frac{\partial H}{\partial p_s} N_s, \qquad q_s \in I_1, \tag{19}$$

where N_s is of the first degree in the momenta. Differentiating (19) twice with respect to p_r, we obtain

$$\frac{\partial g^{rr}}{\partial q_s} \equiv 2 g^{rs} \frac{\partial N_s}{\partial p_r},$$

which proves that (18) still holds when $g^{rs} \equiv 0$. The same properties for the g^{ij} were also found to hold for separable conservative systems (Dall'Acqua 1912). If q_s and q_r are both variables of the second kind, (5*c*) becomes

$$g^{s} \frac{\partial g^{r}}{\partial q_s} \equiv g^{r} M^{(0)}_{sr},$$

or, by (9*b*)

$$g^{s} \frac{\partial g^{r}}{\partial q_s} \equiv g^{r} \frac{g^{s}}{g^{rr}} \frac{\partial g^{rr}}{\partial q_s}.$$

If $g^{s} \not\equiv 0$, we have

$$\frac{\partial g^{r}}{\partial q_s} - \frac{g^{r}}{g^{rr}} \frac{\partial g^{rr}}{\partial q_s} \equiv 0.$$

In the case $g^{s} \equiv 0$, this relation follows immediately from (5*b*), together with (9*a*), (9*b*) and (16). Following the same argument as was used for obtaining (17), we get

$$\frac{\partial}{\partial q_s} \left(\frac{g^{r}}{g^{rr}} \right) \equiv 0, \qquad q_r \in I_2,\ q_s \in I_2,\ r \neq s. \tag{20}$$

We now return to the relations (6), (6*a*) and (6*b*). Differentiation of (6*a*) with respect to p_r yields

$$\frac{\partial g^{rr}}{\partial t} \equiv g^{rr} K^{(0)}_{r}. \tag{21}$$

Comparing this with (6*b*), we see that

$$\frac{\partial g^r}{\partial t}-\frac{g^r}{g^{rr}}\frac{\partial g^{rr}}{\partial t}\equiv 0,$$

and so

$$\frac{\partial}{\partial t}\left(\frac{g^r}{g^{rr}}\right)\equiv 0, \qquad q_r\in I_2. \tag{22}$$

Combining (21) and (6) one can also easily prove that

$$\frac{\partial}{\partial t}\left(\frac{1}{g^{rr}}\frac{\partial H}{\partial p_r}\right)\equiv 0, \qquad q_r\in I_2. \tag{23}$$

Henceforward we shall suppose, without loss of generality, that

$$I_1=\{q_1, q_2, \ldots, q_{\bar{n}}\},$$
$$I_2=\{q_{\bar{n}+1}, q_{\bar{n}+2}, \ldots, q_n\},$$

where $0\leqslant\bar{n}\leqslant n$, with $\bar{n}=0$ and $\bar{n}=n$ corresponding respectively to the cases $I_1=\varnothing$, $I_2=\{q_1, q_2, \ldots, q_n\}$ and $I_1=\{q_1, q_2, \ldots, q_n\}$, $I_2=\varnothing$.

Unless stated otherwise the Latin indices *i, j, k, l, m* will refer to variables of the first kind and the indices *r, s, u* to those of the second kind. Using the same convention as Havas (1975), summation from 1 up to and including $\bar{n}$ will be indicated by Σ^{I} and summation from $\bar{n}+1$ up to and including n by Σ^{II}.

As a last step in this section we are now going to examine the functions N_i $(i=1, 2, \ldots, \bar{n})$. Differentiation of (4) twice with respect to p_i shows after a short calculation, similar to the one we have performed to derive (9), that

$$N_i\equiv\frac{1}{2(g^{ii})^2}\left(2g^{ii}\frac{\partial^2 H}{\partial q_i\,\partial p_i}-\frac{\partial g^{ii}}{\partial q_i}\frac{\partial H}{\partial p_i}\right), \qquad i=1, 2, \ldots, \bar{n}. \tag{24}$$

Taking into account (9) and (18), the identity (5) becomes

$$\frac{\partial H}{\partial p_i}\frac{\partial^2 H}{\partial q_i\,\partial p_r}-\frac{\partial H}{\partial q_i}g^{ir}\equiv 0$$

and, after differentiating twice with respect to p_i, we obtain

$$2g^{ii}\frac{\partial g^{ir}}{\partial q_i}-g^{ir}\frac{\partial g^{ii}}{\partial q_i}\equiv 0, \qquad i=1, 2, \ldots, \bar{n}, \qquad r=\bar{n}+1, \bar{n}+2, \ldots, n. \tag{25}$$

Inserting (25) into (24) we finally arrive at

$$N_i\equiv\sum_j{}^{\mathrm{I}}\,\lambda_{ij}(q, t)p_j+\lambda_i(q, t), \tag{26}$$

where

$$\lambda_{ij}\equiv\frac{1}{g^{ii}}\frac{\partial g^{ij}}{\partial q_i}-\frac{1}{2(g^{ii})^2}\frac{\partial g^{ii}}{\partial q_i}g^{ij}$$

and $\qquad\qquad (i, j=1, 2, \ldots, \bar{n}).$

$$\lambda_i\equiv\frac{1}{g^{ii}}\frac{\partial g^i}{\partial q_i}-\frac{1}{2(g^{ii})^2}\frac{\partial g^{ii}}{\partial q_i}g^i$$

Consequently, the functions N_i are independent of the momenta conjugated to variables of the second kind (which can also be proved indirectly).

The preceding results will now suffice to construct a complete integral of the form (3) for the given Hamilton–Jacobi equation (2) which, by assumption, is known to be separable.

3. General solution

Let us denote the complete integral we are looking for, by

$$\bar{W}(q, c, t) \equiv \bar{W}_0(c, t) + \sum_i^{\mathrm{I}} \bar{W}_i(q_i, c) + \sum_r^{\mathrm{II}} \bar{W}_r(q_r, c).$$

The momenta will then be given by

$$p_j(q_j) = \frac{\mathrm{d}\bar{W}_j}{\mathrm{d}q_j}, \qquad j = 1, 2, \ldots, n. \tag{27}$$

We shall calculate these functions by means of the method introduced by Dall'Acqua (1912). The treatment will clearly differ according to the kind of variable we are dealing with. In the case of a separable Hamilton–Jacobi equation, the following relations are known to hold identically (Levi-Civita 1904, Forbat 1944):

$$\frac{\mathrm{d}}{\mathrm{d}q_j} p_j(q_j) \equiv -\frac{\partial H/\partial q_j}{\partial H/\partial p_j}, \qquad j = 1, 2, \ldots, n,$$

where all the momenta are considered as functions of the corresponding coordinate, given by (27). For the momenta conjugated to a variable of the first kind, we then have by (4) and (26)

$$\frac{\mathrm{d}}{\mathrm{d}q_i} p_i(q_i) \equiv -\sum_j^{\mathrm{I}} \lambda_{ij}(q, t) p_j(q_j) - \lambda_i(q, t), \qquad i = 1, 2, \ldots, \bar{n}. \tag{28}$$

Fixing all variables in this identity, different from q_i, at their initial value ($t = t^0$, $q = q_j^0$) and putting the constants $p_j(q_j^0) = c_j'$, we obtain the following linear ordinary differential equation:

$$\frac{\mathrm{d}p_i}{\mathrm{d}q_i} + \bar{\lambda}_{ii}(q_i) p_i = -\sum_{j \neq i}^{\mathrm{I}} \bar{\lambda}_{ij}(q_i) c_j' - \bar{\lambda}_i(q_i),$$

where

$$\bar{\lambda}_{ij}(q_i) \equiv \lambda_{ij}(q_1^0, \ldots, q_{i-1}^0, q_i, q_{i+1}^0, \ldots, t^0)$$

and

$$\bar{\lambda}_i(q_i) \equiv \lambda_i(q_1^0, \ldots, q_{i-1}^0, q_i, q_{i+1}^0, \ldots, t^0).$$

The solution of this equation can be written as

$$p_i(q_i) = \sum_j^{\mathrm{I}} \phi_{ij}(q_i) c_j' + \psi_i(q_i), \qquad i = 1, 2, \ldots, \bar{n}, \tag{29}$$

where ψ_i and ϕ_{ij} ($j = 1, 2, \ldots, \bar{n}$) are known functions of q_i only. Using the expressions for the λ_{ij} (see (26)) one can calculate the functions ϕ_{ij} explicitly and verify that

$\det(\phi_{ij}) \not\equiv 0$. In order to find a solution for the momenta p_r ($r = \bar{n}+1, \bar{n}+2, \ldots, n$), we return to the Hamilton–Jacobi equation (2). This equation must be fulfilled identically by $\bar{W}(q, c, t)$. Putting $R(t) \equiv \partial\bar{W}/\partial t$ ($\equiv \mathrm{d}\bar{W}_0/\mathrm{d}t$) and using (27), we then have

$$\tfrac{1}{2} \sum_{i,j=1}^{n} g^{ij}(q, t) p_i(q_i) p_j(q_j) + \sum_{i=1}^{n} g^i(q, t) p_i(q_i) + \tfrac{1}{2} g^0(q, t) + V(q, t) + R(t) \equiv 0. \tag{30}$$

Fixing all variables, excepted q_r (for some $r \in \{\bar{n}+1, \ldots, n\}$), at their initial value, and taking into account (16), we get, after re-arrangement of the terms,

$$g_r^{rr} p_r^2 + 2\left(\sum_i^{\mathrm{I}} g_r^{ir} c_i' + g_r^r\right) p_r + \sum_{i,j}^{\mathrm{I}} g_r^{ij} c_i' c_j' + \phi_r + 2 \sum_i^{\mathrm{I}} g_r^i c_i' + g_r^0 + 2V_r + 2R_0 = 0, \tag{31}$$

where

$$R_0 = R(t^0),$$

$$\phi_r \equiv 2 \sum_i^{\mathrm{I}} \sum_{s \neq r}^{\mathrm{II}} g_r^{is} c_i' c_s' + \sum_{s \neq r}^{\mathrm{II}} g_r^{ss} c_s'^2 + 2 \sum_{s \neq r}^{\mathrm{II}} g_r^s c_s',$$

and

$$c_j' = p_j(q_j^0) \qquad \text{for } j = 1, 2, \ldots, n.$$

The lower index r indicates that the corresponding functions depend on q_r only.

We shall now transform the expression for ϕ_r. From (17) we have

$$\frac{\partial}{\partial q_r}\left(\frac{1}{g^{ss}} \frac{\partial H}{\partial p_s}\right) \equiv 0, \qquad r, s = \bar{n}+1, \bar{n}+2, \ldots, n, \qquad r \neq s.$$

Differentation with respect to p_i (for some $i \in \{1, 2, \ldots, \bar{n}\}$) yields

$$\frac{\partial}{\partial q_r}\left(\frac{g^{is}}{g^{ss}}\right) \equiv 0,$$

and, in particular

$$\frac{\partial}{\partial q_r}\left(\frac{g_r^{is}}{g_r^{ss}}\right) \equiv 0.$$

Integration over q_r (between q_r^0 and q_r) gives immediately

$$\frac{g_r^{is}}{g_r^{ss}} \equiv \frac{g_0^{is}}{g_0^{ss}} \qquad \text{or} \qquad g_r^{is} \equiv g_r^{ss} \frac{g_0^{is}}{g_0^{ss}}, \tag{32}$$

where g_0^{is} and g_0^{ss} are constants†. Similarly one obtains from (20)

$$g_r^s \equiv g_r^{ss} \left(\frac{g_0^s}{g_0^{ss}}\right). \tag{33}$$

With (32) and (33), ϕ_r now becomes

$$\phi_r \equiv \sum_{s \neq r}^{\mathrm{II}} g_r^{ss} \left(2 \sum_i^{\mathrm{I}} \frac{g_0^{is}}{g_0^{ss}} c_i' c_s' + c_s'^2 + 2 \frac{g_0^s}{g_0^{ss}} c_s'\right).$$

---

† We put $g^{ij}(q_1^0, \ldots, q_n^0, t^0) = g_0^{ij}$ and $g^i(q_1^0, \ldots, q_n^0, t^0) = g_0^i$ for $i, j = 1, 2, \ldots, n$.

The expression in large parentheses is a constant for each s. Putting

$$c''_s = 2\sum_i^{\mathrm{I}} \frac{g_0^{is}}{g_0^{ss}} c'_i c'_s + c'^2_s + 2\frac{g_0^s}{g_0^{ss}} c'_s \qquad \text{for all } s = \bar{n}+1, \bar{n}+2, \ldots, n,$$

we have

$$\phi_r \equiv \sum_{s \neq r}^{\mathrm{II}} g_r^{ss} c''_s . \tag{34}$$

One can easily verify that the constants $c'_1, \ldots, c'_{\bar{n}}, c''_{\bar{n}+1}, \ldots, c''_n$ are independent. Finally we still have to determine the constant R_0. For that purpose we fix all variables in (30) at their initial value:

$$\tfrac{1}{2} \sum_{i,j=1}^{n} g_0^{ij} c'_i c'_j + \sum_{i=1}^{n} g_0^i c'_i + \tfrac{1}{2} g_0^0 + V_0 + R_0 = 0,$$

and after an analogous calculation to the one we have just performed to derive (34):

$$\sum_{i,j}^{\mathrm{I}} g_0^{ij} c'_i c'_j + \sum_s^{\mathrm{II}} g_0^{ss} c''_s + 2\sum_i^{\mathrm{I}} g_0^i c'_i + g_0^0 + 2V_0 + 2R_0 = 0.$$

The constant term V_0, arising from the potential energy, being arbitrary, we put $V_0 = -\frac{1}{2} g_0^0$, so that

$$2R_0 = -\sum_{i,j}^{\mathrm{I}} g_0^{ij} c'_i c'_j - \sum_s^{\mathrm{II}} g_0^{ss} c''_s - 2\sum_i^{\mathrm{I}} g_0^i c'_i . \tag{35}$$

Substituting (34) and (35) into (31), the quadratic equation in p_r becomes

$$g_r^{rr} p_r^2 + 2\left(\sum_i^{\mathrm{I}} g_r^{ir} c'_i + g_r^r\right) p_r + \sum_{i,j}^{\mathrm{I}} (g_r^{ij} - g_0^{ij}) c'_i c'_j + \sum_s^{\mathrm{II}} [g_r^{ss}(1-\delta_{sr}) - g_0^{ss}] c''_s$$
$$+ 2\sum_i^{\mathrm{I}} (g_r^i - g_0^i) c'_i + g_r^0 + 2V_r = 0,$$

(with δ_{sr} the Kronecker delta). An elementary calculation shows that the solution is of the form

$$p_r(q_r) = \sum_j^{\mathrm{I}} f_r^j(q_r) c'_j + \psi_r(q_r) \pm \left(\sum_{k,l}^{\mathrm{I}} F_r^{kl}(q_r) c'_k c'_l + \sum_j^{\mathrm{I}} h_r^j(q_r) c'_j + \sum_s^{\mathrm{II}} \phi_{rs}(q_r) c''_s - 2u_r(q_r)\right)^{1/2},$$
$$r = \bar{n}+1, \bar{n}+2, \ldots, n, \tag{36}$$

where F_r^{kl}, h_r^j, f_r^j, ϕ_{rs}, u_r and ψ_r are known functions of q_r only and $F_r^{kl}(q_r) \equiv F_r^{lk}(q_r)$ ($j, k, l = 1, 2, \ldots, \bar{n}$; $s = \bar{n}+1, \bar{n}+2, \ldots, n$). Again one can easily check that $\det(\phi_{rs}) \neq 0$.

It only remains for us now to calculate $R(t)$. Fixing all variables, except the time t, at their initial value, we get from the identity (30)

$$2R(t) = -\sum_{i,j}^{\mathrm{I}} g_t^{ij} c'_i c'_j - 2\sum_i^{\mathrm{I}} \sum_s^{\mathrm{II}} g_t^{is} c'_i c'_s - \sum_s^{\mathrm{II}} g_t^{ss} c'^2_s - 2\sum_s^{\mathrm{II}} g_t^s c'_s - 2\sum_i^{\mathrm{I}} g_t^i c'_i - g_t^0 - 2V_t,$$

where the lower index t indicates the time dependence (for instance: $V_t \equiv V(q_1^0, \ldots, q_n^0, t)$). Following an analogous calculation to the one used for obtaining (32) and (33), one can derive from (22) and (23) that

$$g_t^s \equiv g_t^{ss}\left(\frac{g_0^s}{g_0^{ss}}\right),$$

and

$$g_t^{is} \equiv g_t^{ss}\left(\frac{g_0^{is}}{g_0^{ss}}\right)$$

must hold. Inserting this into the expression for $R(t)$, we arrive at

$$R(t) = -\frac{1}{2}\left[\sum_{i,j}^{\mathrm{I}} g_t^{ij} c_i' c_j' + \sum_s^{\mathrm{II}} g_t^{ss}\left(2\sum_i^{\mathrm{I}} \frac{g_0^{is}}{g_0^{ss}} c_i' c_s' + c_s'^2 + 2\frac{g_0^s}{g_0^{ss}} c_s'\right) + 2\sum_i^{\mathrm{I}} g_t^i c_i' + g_t^0 + 2V_t\right].$$

We notice that the coefficient of g_t^{ss} is just the constant c_s'', and so, after changing our notations, $R(t)$ can be written as

$$R(t) = -\tfrac{1}{2}\left(\sum_{k,l}^{\mathrm{I}} G^{kl}(t) c_k' c_l' + \sum_s^{\mathrm{II}} l^s(t) c_s'' + \sum_j^{\mathrm{I}} k^j(t) c_j' + v(t)\right) \tag{37}$$

where G^{kl}, k^j, l^s and v are known functions of the time only and $G^{kl}(t) \equiv G^{lk}(t)$ ($j, k, l = 1, 2, \ldots, \bar{n}$; $s = \bar{n}+1, \bar{n}+2, \ldots, n$). Henceforward we shall denote the constants by c_j ($j = 1, 2, \ldots, n$) such that $c_i = c_i'$ for $i = 1, 2, \ldots, \bar{n}$ and $c_s = c_s''$ for $s = \bar{n}+1, \bar{n}+2, \ldots, n$. In what precedes we have proved that whenever the Hamilton–Jacobi equation (2) is separable, there exists a solution

$$\bar{W}(q, c, t) \equiv \bar{W}_0(c, t) + \sum_{i=1}^{\bar{n}} \bar{W}_i(q_i, c) + \sum_{r=\bar{n}+1}^{n} \bar{W}_r(q_r, c) \qquad \text{for some } \bar{n} \in \{0, 1, 2, \ldots, n\},$$

satisfying the following ordinary differential equations:

$$\frac{\mathrm{d}\bar{W}_i}{\mathrm{d}q_i} = \sum_j^{\mathrm{I}} \phi_{ij}(q_i) c_j + \psi_i(q_i), \tag{29a}$$

$$\frac{\mathrm{d}\bar{W}_r}{\mathrm{d}q_r} = \sum_j^{\mathrm{I}} f_r^j(q_r) c_j + \psi_r(q_r) \pm \left(\sum_{k,l}^{\mathrm{I}} F_r^{kl}(q_r) c_k c_l + \sum_j^{\mathrm{I}} h_r^j(q_r) c_j + \sum_s^{\mathrm{II}} \phi_{rs}(q_r) c_s - 2u_r(q_r)\right)^{1/2}, \tag{36a}$$

$$\frac{\mathrm{d}\bar{W}_0}{\mathrm{d}t} = -\tfrac{1}{2}\left(\sum_{k,l}^{\mathrm{I}} G^{kl}(t) c_k c_l + \sum_s^{\mathrm{II}} l^s(t) c_s + \sum_j^{\mathrm{I}} k^j(t) c_j + v(t)\right), \tag{37a}$$

where $c_1, c_2, \ldots, c_n$ are n real independent arbitrary constants. The complete integral $\bar{W}$ is then given by

$$\begin{aligned}\bar{W}(q, c, t) = {} & \sum_i^{\mathrm{I}} \int \left(\sum_j^{\mathrm{I}} \phi_{ij} c_j + \psi_i\right) \mathrm{d}q_i \\ & + \sum_r^{\mathrm{II}} \int \left[\sum_j^{\mathrm{I}} f_r^j c_j + \psi_r \pm \left(\sum_{k,l}^{\mathrm{I}} F_r^{kl} c_k c_l + \sum_j^{\mathrm{I}} h_r^j c_j + \sum_s^{\mathrm{II}} \phi_{rs} c_s - 2u_r\right)^{1/2}\right] \mathrm{d}q_r \\ & - \tfrac{1}{2} \int \left(\sum_{k,l}^{\mathrm{I}} G^{kl} c_k c_l + \sum_s^{\mathrm{II}} l^s c_s + \sum_j^{\mathrm{I}} k^j c_j + v\right) \mathrm{d}t.\end{aligned} \tag{38}$$

One can easily show that $\det(\partial^2 \bar{W}/\partial q_i\, \partial c_j)_{i,j=1,2,\ldots,n} \neq 0$ since neither $\det(\phi_{kl})$ nor $\det(\phi_{rs})$ vanish identically ($k, l = 1, 2, \ldots, \bar{n}$; $r, s = \bar{n}+1, \bar{n}+2, \ldots, n$).

Next we shall prove that every function of the form (38) is a complete integral of a partial differential equation of the Hamilton–Jacobi type. Suppose we are given the following set of arbitrary real continuous functions of a single variable each: $\phi_{ij}(q_i)$, $\phi_{rs}(q_r)$, $\psi_i(q_i)$, $\psi_r(q_r)$, $f^i_r(q_r)$, $h^i_r(q_r)$, $u_r(q_r)$, $F^{kl}_r(q_r)$, $G^{kl}(t)$, $l^s(t)$, $k^i(t)$ and $v(t)$, with $i, j, k, l = 1, 2, \ldots, \bar{n}$ and $r, s = \bar{n}+1, \bar{n}+2, \ldots, n$ for some $\bar{n} \in \{0, 1, \ldots, n\}$, such that $F^{kl}_r(q_r) \equiv F^{lk}_r(q_r)$, $G^{kl}(t) \equiv G^{lk}(t)$, $\det(\phi_{ij}) \neq 0$ and $\det(\phi_{rs}) \neq 0$. Furthermore we are given a set of n real arbitrary independent constants $c_1, c_2, \ldots, c_n$. Consider now the function

$$\bar{W} \equiv \bar{W}_0(c, t) + \sum_i^{\mathrm{I}} \bar{W}_i(q_i, c) + \sum_r^{\mathrm{II}} \bar{W}_r(q_r, c)$$

defined by (38), where $\bar{W}_i$, $\bar{W}_r$ and $\bar{W}_0$ respectively satisfy the equations (29*a*), (36*a*) and (37*a*).

We introduce the following notations: $\det(\phi_{ij}) \equiv \Phi_{\mathrm{I}}(\neq 0)$ and $\det(\phi_{rs}) \equiv \Phi_{\mathrm{II}}(\neq 0)$. The cofactors of ϕ_{ij} and ϕ_{rs} will be respectively denoted by Φ_{ij} and Φ_{rs}. We then have the well known relations

$$\sum_{i=1}^{\bar{n}} \phi_{ij}\Phi_{im} = \delta_{jm}\Phi_{\mathrm{I}}, \qquad \sum_{r=\bar{n}+1}^{n} \phi_{rs}\Phi_{ru} = \delta_{su}\Phi_{\mathrm{II}},$$

with $m = 1, 2, \ldots, \bar{n}$ and $u = \bar{n}+1, \bar{n}+2, \ldots, n$. If (ϕ_{ij}) or (ϕ_{rs}) consist of a single element only, we put the corresponding cofactor identical to 1. Elimination of the constants from equations (29*a*), (36*a*) and (37*a*) will lead to the partial differential equation of which $\bar{W}$ is a complete integral. Multiplying (29*a*) by $\Phi_{im}/\Phi_{\mathrm{I}}$ and summing over i, we obtain after re-arrangement

$$\sum_i^{\mathrm{I}} \left(\frac{\mathrm{d}\bar{W}_i}{\mathrm{d}q_i} - \psi_i\right)\frac{\Phi_{im}}{\Phi_{\mathrm{I}}} = c_m, \qquad m = 1, 2, \ldots, \bar{n}. \tag{39}$$

The constants $c_r (r = \bar{n}+1, \bar{n}+2, \ldots, n)$ can be calculated from (36*a*). After re-arrangement of the terms and squaring, we multiply this equation by $\Phi_{ru}/\Phi_{\mathrm{II}}$ and sum over r. Using (39) we then find

$$\begin{aligned}
&\sum_r^{\mathrm{II}} \left[\frac{\mathrm{d}\bar{W}_r}{\mathrm{d}q_r} - \sum_{i,j}^{\mathrm{I}} f^j_r\left(\frac{\mathrm{d}\bar{W}_i}{\mathrm{d}q_i} - \psi_i\right)\frac{\Phi_{ij}}{\Phi_{\mathrm{I}}} - \psi_r\right]^2 \frac{\Phi_{ru}}{\Phi_{\mathrm{II}}} - \sum_r^{\mathrm{II}} \sum_{\substack{i,j\\k,l}}^{\mathrm{I}} \left[F^{kl}_r\left(\frac{\mathrm{d}\bar{W}_i}{\mathrm{d}q_i} - \psi_i\right)\left(\frac{\mathrm{d}\bar{W}_j}{\mathrm{d}q_j} - \psi_j\right)\frac{\Phi_{ik}\Phi_{jl}}{\Phi_{\mathrm{I}}^2}\right]\frac{\Phi_{ru}}{\Phi_{\mathrm{II}}} \\
&\qquad - \sum_r^{\mathrm{II}} \sum_{i,j}^{\mathrm{I}} \left[h^j_r\left(\frac{\mathrm{d}\bar{W}_i}{\mathrm{d}q_i} - \psi_i\right)\frac{\Phi_{ij}}{\Phi_{\mathrm{I}}}\right]\frac{\Phi_{ru}}{\Phi_{\mathrm{II}}} + 2\sum_r^{\mathrm{II}} \frac{u_r \Phi_{ru}}{\Phi_{\mathrm{II}}} = c_u, \\
&\qquad u = \bar{n}+1, \bar{n}+2, \ldots, n.
\end{aligned} \tag{40}$$

Substitution of (39) and (40) into (37*a*) finally gives

$$\begin{aligned}
&\sum_{\substack{i,j\\k,l}}^{\mathrm{I}} G^{kl}(t)\left(\frac{\mathrm{d}\bar{W}_i}{\mathrm{d}q_i} - \psi_i\right)\left(\frac{\mathrm{d}\bar{W}_j}{\mathrm{d}q_j} - \psi_j\right)\frac{\Phi_{ik}\Phi_{jl}}{\Phi_{\mathrm{I}}^2} + \sum_{i,j}^{\mathrm{I}} k^j(t)\left(\frac{\mathrm{d}\bar{W}_i}{\mathrm{d}q_i} - \psi_i\right)\frac{\Phi_{ij}}{\Phi_{\mathrm{I}}} \\
&\qquad + \sum_{r,s}^{\mathrm{II}} \frac{l^s(t)\Phi_{rs}}{\Phi_{\mathrm{II}}}\left[\frac{\mathrm{d}\bar{W}_r}{\mathrm{d}q_r} - \sum_{i,j}^{\mathrm{I}} f^j_r\left(\frac{\mathrm{d}\bar{W}_i}{\mathrm{d}q_i} - \psi_i\right)\frac{\Phi_{ij}}{\Phi_{\mathrm{I}}} - \psi_r\right]^2
\end{aligned}$$

$$
\begin{aligned}
&-\sum_{r,s}^{\mathrm{II}} \frac{l^s(t)\Phi_{rs}}{\Phi_{\mathrm{II}}}\left[\sum_{\substack{i,j\\k,l}}^{\mathrm{I}} F_r^{kl}\left(\frac{\mathrm{d}\bar{W}_i}{\mathrm{d}q_i}-\psi_i\right)\left(\frac{\mathrm{d}\bar{W}_j}{\mathrm{d}q_j}-\psi_j\right)\frac{\Phi_{ik}\Phi_{jl}}{\Phi_{\mathrm{I}}^2}\right]\\
&-\sum_{r,s}^{\mathrm{II}} \frac{l^s(t)\Phi_{rs}}{\Phi_{\mathrm{II}}}\left[\sum_{i,j}^{\mathrm{I}} h_r^j\left(\frac{\mathrm{d}\bar{W}_i}{\mathrm{d}q_i}-\psi_i\right)\frac{\Phi_{ij}}{\Phi_{\mathrm{I}}}\right]\\
&+2\sum_{r,s}^{\mathrm{II}} \frac{l^s(t)\Phi_{rs}}{\Phi_{\mathrm{II}}}u_r+v(t)+2\frac{\mathrm{d}\bar{W}_0}{\mathrm{d}t}=0.
\end{aligned}
\tag{41}
$$

Working out the left-hand side and re-arranging the terms, we can ascertain that $\bar{W}$ is indeed a complete integral of a partial differential equation of the type (2). Since $\bar{n}$ may be any integer from 0 up to and including n there are consequently $n+1$ types of separable systems with n degrees of freedom.

Comparing the partial differential equation corresponding to (41) with (2) and putting $\Phi_{ij}/\Phi_{\mathrm{I}} \equiv \eta_{ij}$ $(i, j, = 1, 2, \ldots, \bar{n})$ and $\Phi_{rs}/\Phi_{\mathrm{II}} \equiv \eta_{rs}$ $(r, s = \bar{n}+1, \bar{n}+2, \ldots, n)$, we arrive at the following expressions:

$$
\begin{aligned}
&g^{rr} \equiv \sum_{s}^{\mathrm{II}} l^s(t)\eta_{rs},\\
&g^{rs} \equiv 0, \qquad r \neq s,\\
&g^{ij} \equiv \sum_{k,l}^{\mathrm{I}} \left(G^{kl}(t)-\sum_{r}^{\mathrm{II}} (F_r^{kl}-f_r^k f_r^l)g^{rr}\right)\eta_{ik}\eta_{jl},\\
&g^{ir} \equiv -\sum_{j}^{\mathrm{I}} f_r^j g^{rr}\eta_{ij},\\
&g^{r} \equiv \left(\sum_{i,j}^{\mathrm{I}} f_r^j\psi_i\eta_{ij}-\psi_r\right)g^{rr},\\
&g^{i} \equiv \tfrac{1}{2}\sum_{j}^{\mathrm{I}} \left(k^j(t)+\sum_{r}^{\mathrm{II}} (2f_r^j\psi_r-h_r^j)g^{rr}\right)\eta_{ij}-\sum_{j,k,l}^{\mathrm{I}} \left(G^{kl}(t)-\sum_{r}^{\mathrm{II}} (F_r^{kl}-f_r^k f_r^l)g^{rr}\right)\psi_j\eta_{ik}\eta_{jl},\\
&\tfrac{1}{2}g^0+V \equiv \tfrac{1}{2}\sum_{\substack{i,j\\k,l}}^{\mathrm{I}} \left(G^{kl}(t)-\sum_{r}^{\mathrm{II}} (F_r^{kl}-f_r^k f_r^l)g^{rr}\right)\psi_i\psi_j\eta_{ik}\eta_{jl}\\
&\qquad -\tfrac{1}{2}\sum_{i,j}^{\mathrm{I}} \left(k^j(t)+\sum_{r}^{\mathrm{II}} (2f_r^j\psi_r-h_r^j)g^{rr}\right)\psi_i\eta_{ij}+\tfrac{1}{2}\sum_{r}^{\mathrm{II}} (\psi_r^2+2u_r)g^{rr}+\tfrac{1}{2}v(t),\\
&i, j = 1, 2, \ldots, \bar{n}, \qquad r, s = \bar{n}+1, \bar{n}+2, \ldots, n,
\end{aligned}
\tag{42}
$$

where all the functions on the right-hand sides should be interpreted as before.

From the preceding it follows that (42) represents a necessary condition for the separability of equation (2). Conversely, suppose we have a Hamiltonian of form (1), such that (42) is satisfied for some $\bar{n} \in \{0, 1, \ldots, n\}$. We can then verify that the function $\bar{W}$, defined by (38), is a complete integral of the corresponding Hamilton–Jacobi equation. We can therefore state the following theorem.

Theorem. Suppose we are given a Hamiltonian of type (1). The necessary and sufficient conditions for the corresponding Hamilton–Jacobi equation to be separable are that the functions g^{ij}, $g^i (i, j = 1, 2, \ldots, n)$ and $\frac{1}{2}g^0 + V$ (eventually after renumbering the variables) can be written in the form (42) for some $\bar{n} \in \{0, 1, \ldots, n\}$, where:

(i) $G^{kl}(t)$, $l^s(t)$, $k^j(t)$, $v(t)$, $F_r^{kl}(q_r)$, $f_r^j(q_r)$, $h_r^j(q_r)$, $\psi_i(q_i)$, $\psi_r(q_r)$ and $u_r(q_r)$ are arbitrary, real continuous functions of one variable each;

(ii) $G^{kl}(t) \equiv G^{lk}(t)$, $F_r^{kl}(q_r) \equiv F_r^{lk}(q_r)$;

(iii) two sets of real continuous functions $\phi_{ij}(q_i)(i, j = 1, 2, \ldots, \bar{n})$ and $\phi_{rs}(q_r)$ $(r, s = \bar{n} + 1, \bar{n} + 2, \ldots, n)$ exist, each with non-vanishing determinant, such that

$$\sum_i^{\mathrm{I}} \phi_{ij}\eta_{ik} = \delta_{jk} \quad \text{and} \quad \sum_r^{\mathrm{II}} \phi_{rs}\eta_{ru} = \delta_{su}.$$

4. Remarks

(i) The expressions in (42) are in accordance with those obtained by Iarov-Iarovoi (1963).

For ‘natural’ conservative systems (i.e. conservative systems having a Hamiltonian without linear terms in the momenta) we recover the results derived by Havas (1975), taking into account, however, the modified significance of the constants appearing in the complete integral.

The above results are further also applicable to general, non-natural conservative systems: e.g. the problem of the spinning top, for which the Hamilton–Jacobi equation is separable (Pars 1965).

(ii) For completeness it may be noticed that from a mechanical point of view, the additive function of time $v(t)$, appearing in the Hamiltonian, is superfluous. This follows immediately from the equations of motion, where such a term vanishes. Therefore, the function $v(t)$ may be omitted. This can be justified by observing that it is allowed to modify the potential energy (by adding an arbitrary continuous function of time) in order to obtain $V(q_1^0, \ldots, q_n^0, t) = -\frac{1}{2}g^0(q_1^0, \ldots, q_n^0, t)$, yielding $v(t) = 2V_t + g_t^0 = 0$ (see (37)).

(iii) Let us now consider the special case of a time-dependent Hamiltonian for which $H_1 \equiv 0$ and H_0 is supposed to depend on all the coordinates. The property $H_1 \equiv 0$ is clearly equivalent to $g^j \equiv 0$ for all $j = 1, 2, \ldots, n$. If the Hamilton–Jacobi equation for such system is separable, it then follows from (4c) that

$$\frac{\partial H_0}{\partial q_i} \equiv 0,$$

for $i = 1, 2, \ldots, \bar{n}$. Consequently, since H_0 depends on all the coordinates, there may be no variables of the first kind, i.e. $\bar{n} = 0$. By means of the preceding results one can now readily verify that the Hamiltonian must be of the form

$$H \equiv \frac{1}{2} \sum_{r=1}^{n} g^{rr} p_r^2 + \frac{1}{2}g^0 + V,$$

with

$$g^{rr} \equiv \sum_{s=1}^{n} l^s(t)\eta_{rs}, \qquad \frac{1}{2}g^0 + V \equiv \sum_{r=1}^{n} u_r g^{rr} + \frac{1}{2}v(t).$$

The complete integral will then be given by

$$\bar{W} = \sum_{r=1}^{n} \int \pm \left(\sum_{s=1}^{n} \phi_{rs} c_s - 2u_r \right)^{1/2} \mathrm{d}q_r - \tfrac{1}{2} \int \left(\sum_{s=1}^{n} l^s(t) c_s + v(t) \right) \mathrm{d}t.$$

This is precisely the solution found by Forbát (1944).

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References

Agostinelli C 1936–7 *Atti R. Ist. Veneto Sci.* **46** 151–61
——— 1937 *Memorie R. Accad. Sci. Torino* **69** 3–54
Burgatti P 1911 *Atti. Accad. Naz. Lincei Rc.* **20** 108–11
Dall'Acqua F A 1912 *Rc. Circ. Mat. Palermo* **33** 341–51
Forbat N 1944 *Bull. Acad. R. Belg. Cl. Sci.* **30** 462–73
Gantmacher F 1970 *Lectures in Analytical Mechanics* (Moscow: Mir Publishers) pp 67–8, 47
Havas P 1975 *J. Math. Phys.* **16** 1461–8
Iarov-Iarovoi M S 1963 *Prikl. Mat. Mekh.* **27** 973–87 (1964 *J. Appl. Math. Mech.-PMM* **27** 1499–520)
Levi-Civita T 1904 *Math. Annln* **59** 383–97
Pars L A 1965 *A Treatise on Analytical Dynamics* (London: Heinemann) p 348